Author instructions for the presentation and analysis of paraxial and thin lens refractive data.

The following website can be used for calculations http://OphthaCalc.co.uk/

Excel files to facilitate calculations and significance testing of refractive data will be made available upon request to **info.bmjophth@bmj.com**

Presentation

Paraxial refractive data should be presented in spherocylinder form, Sphere/Cylinder x Axis (S/CxA) using 2 decimal places for the sphere and cylinder and no decimal places for the axis. The sphere and cylinder should be presented as a compound number, for example, -1.00/+1.50 x 170, not as individual terms.[1] If the spherical equivalent (SE) is to be used, it should be presented in parentheses after the compound refractive error, for example, -1.00/+1.50 x 170 (SE -0.25). The spherical equivalent should not be presented in isolation.

Analysis

Paraxial refractive data in the form S/CxA should be transformed into a in the 2 x2 matrix using Long's method [2] as follows.

$S/C_A = \begin{matrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{matrix}$ where, in the 2 x2 matrix, the cell in the first row and first column is denoted by f_{11} and the cell in the first row and second column is denoted by f_{12} an so on. Long[2] showed that refractive data can be transformed into 4 independent components given by,

$f_{11} = S + C\sin^2 A$, $f_{12} = -C\sin A\cos A$ $f_{21} = -C\sin A\cos A$ and $f_{22} = S + C\cos^2 A$.

Therefore,

$$S/C_A = \begin{matrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{matrix} = \begin{matrix} S + C\sin^2 A & -C\sin A\cos A \\ -C\sin A\cos A & S + C\cos^2 A \end{matrix}.$$

For a thin lens,[3] $f_{12} = f_{21}$ so that,

$$S/C_A = [f_{11} \quad f_{12} \quad f_{22}] = [S + C\sin^2 A \quad -C\sin A\cos A \quad S + C\cos^2 A].$$

For example,

$$+1/+2_{x75} = [1 + 2\sin^2 75 \quad -2\sin_{75}\cos_{75} \quad 1 + 2\cos^2 75] = 2.87 \quad -0.50 \quad 1.13$$

Differences between refractive data [4,5,6] should be calculated with the data in the form of

$$S/C_A = \begin{matrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{matrix}$$

For example, if the post-operative refractive error is $+1/+2_{x75}$

$$+1/+2_{x75} = [1 + 2\sin^2 75 \quad -2\sin_{75}\cos_{75} \quad 1 + 2\cos^2 75] = 2.87 \quad -0.50 \quad 1.13$$

and the target or intended outcome is $0/+0.5_{x150}$

$$0/+0.5_{x150} = [0 + 0.5\sin^2 150 \quad -0.5\sin_{150}\cos_{150} \quad 0 + 0.5\cos^2 150] = 0.13 \quad 0.22 \quad 0.3,$$

Then to calculate the difference ΔF between the Intended (target or predicted) and post-operative refractive outcome we have

$$\Delta F = [f_{11} \quad f_{12} \quad f_{22}]_{Post\,op} - [f_{11} \quad f_{12} \quad f_{22}]_{Target(Intended)}$$

For example, following cataract surgery[1]

$$\Delta F_{11} = f_{11Post\,op} - f_{11Target} = 0.13 - 2.87 = -2.74$$

$$\Delta F_{12} = f_{12Post\,op} - f_{12Target} = 0.22 - -0.50 = 0.72$$

$$\Delta F_{22} = f_{22Post\,op} - f_{22Target} = 0.38 - 1.13 = -0.76$$

$$\Delta F = [-2.74 \quad 0.72 \quad -0.76]$$

This difference $[-2.74 \quad 0.72 \quad -0.76]$ is transposed into S/C_A using Keating's[7] or other methods'[3,4,5], to give +2.97+2.45x162.

Statistical analysis

For datasets the refractive data should be transformed into $S/C_A = \begin{matrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{matrix}$ and used in this format to calculate descriptive statistics such as mean, median, standard deviations, confidence intervals. Statistical tests should be applied to the data in this format. The data can then be transformed back into S/CxA to present the mean, median and confidence intervals.

Using Long's formalism and the methods of Harris,[8] and Kaye and Harris,[4] allows one to calculate the total, mean, standard deviation (SD) upper (UCI) and lower (LCI) confidence intervals, standard error of the mean (SEM). For example,

If the difference (ΔF) between Intended (predicted or target) and actual post-operative cataract outcome[6] is ΔF

$$\Delta F = \begin{bmatrix} f_{11pre} - f_{11post} & f_{12pre} - f_{12post} \\ f_{12pre} - f_{12post} & f_{22pre} - f_{22post} \end{bmatrix}$$

Then descriptive statistics for the mean would be

$$\Delta F_{Mean} = \frac{1}{n}\sum_{1}^{n}\Delta F = \frac{1}{n}\begin{bmatrix} \sum(f_{11pre} - f_{11post}) & \sum(f_{12pre} - f_{12post}) \\ \sum(f_{12pre} - f_{12post}) & \sum(f_{22pre} - f_{22post}) \end{bmatrix}$$

and one can use the method of Harris for statistical analysis and hypothesis tests [8,9] to determine whether the mean differs from zero or a selected value ΔF_{Mean} = 0 or ΔF_{Mean} =1D etc.

Astigmatism (paraxial)

Although it is of practical value to relate the magnitude and direction of the individual patient's cylindrical error with the magnitude of their astigmatism, this approach is analytically problematic because a cylinder contains both spherical and an astigmatic components and as such is not restricted to an astigmatic refractive power.[1,9,10]

At present a Jackson cross cylinder (JCC) is the only true (it has no spherical component) paraxial astigmatic power.[10] If the spherical equivalent (also designated M as described by Thibos et al)[11], $SE = S + \frac{C}{2}$ or in matrix form, $M = \frac{f_{11} + f_{22}}{2}$ then, a JCC is

$$JCC = S/CxA - SE = \begin{bmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{bmatrix} - \begin{bmatrix} \frac{f_{11}+f_{22}}{2} & 0 \\ 0 & \frac{f_{11}+f_{22}}{2} \end{bmatrix}$$

$$JCC = \begin{bmatrix} \frac{f_{11}-f_{22}}{2} & f_{12} \\ f_{12} & \frac{f_{22}-f_{11}}{2} \end{bmatrix} = \frac{C}{2}\begin{bmatrix} -\cos 2\alpha & -\sin 2\alpha \\ -\sin 2\alpha & \cos 2\alpha \end{bmatrix}$$

Example. If a patient's refractive error is -1.00/+1.50 x 45 the spherical equivalent (SE) is -0.25, so that the $JCC = S/CxA - SE = -1/+1.50x45 - (-0.25) = -0.75/+1.50x45$ or in matrix form

$$JCC = \frac{1.50}{2}\begin{bmatrix} -\cos 90 & -\sin 90 \\ -\sin 90 & \cos 90 \end{bmatrix} = 0.75\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

References

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